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Existence of radial solutions for an asymptotically linear p-Laplacian problem

Jorge Cossio *, Sigifredo Herrón

Escuela de Matemáticas, Universidad Nacional de Colombia, Apartado Aéreo 3840, Medellín, Colombia

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ABSTRACT

We prove that an asymptotically linear Dirichlet problem which involves the p-Laplacian operator has multiple radial solutions when the nonlinearity has a positive zero and the range of the 'p-derivative' of the nonlinearity includes at least the first j radial eigenvalues of the p-Laplacian operator. The main tools that we use are a uniqueness result for the p-Laplacian operator and bifurcation theory.

1. Introduction

In this paper we study the nonlinear Dirichlet problem

$$\begin{cases} \Delta_p u + g(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega = \{x \in \mathbb{R}^N\colon\ \|x\| < 1\}$ is the unit ball in $\mathbb{R}^N$, $N \geqslant 2$, $g \in C^1(\mathbb{R})$, $g(0) = 0$ and Δ_p is the p-Laplacian operator defined by

$$\Delta_p u := \operatorname{div}\bigl(|\nabla u|^{p-2}\nabla u\bigr), \quad p > 1.$$

Let us define $\varphi_p(t) := |t|^{p-2}t$ for $t \neq 0$ and $\varphi_p(0) = 0$. Let $\{\mu_k(p)\}_{k\in\mathbb{N}}$ be the sequence of radial eigenvalues for the problem

$$\begin{cases} \bigl(r^{N-1}\varphi_p(u')\bigr)' + \mu r^{N-1}\varphi_p(u) = 0, & 0 < r < 1, \\ u'(0) = 0 = u(1). \end{cases} \tag{2}$$

We assume the following hypotheses on the nonlinearity g:

(g_1) There are positive numbers β_1, β_2 such that $g(\beta_1) = 0$ and

$$\beta_2 = \inf\bigl\{t \in [\beta_1, \infty)\colon\ \forall s > t,\ g(s) > 0\bigr\}.$$

(g_2) For some $j \in \mathbb{N}$, $\mu_j(p) < \lim_{|t|\to\infty} \frac{g(t)}{\varphi_p(t)} =: \lambda_\infty \in \mathbb{R}$.

(g_3) $\lim_{t\to 0} \frac{g'(t)}{\varphi_p'(t)} = \lambda_\infty$.

* Corresponding author.
E-mail addresses: jcossio@unalmed.edu.co (J. Cossio), sherron@unalmed.edu.co (S. Herrón).

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(g_4) There is a constant $C > 0$ such that

$$\limsup_{t\to\beta_i}\left|\frac{g(t)}{\varphi_p(t-\beta_i)}\right| \leqslant C \quad \text{for } i = 1, 2.$$

Because of (g_2), problem (1) is called p-asymptotically linear at infinity. Our goal is to study the existence of radial solutions to (1), and to extend a result due to A. Castro and J. Cossio in [1] for the semilinear case, that is when $p = 2$.

The main result is the following:

Theorem 1.1. *Let us assume $p \geqslant 2$. Under the hypotheses (g_1)–(g_4), problem (1) has at least $4j - 1$ radially symmetric solutions.*

In order to show the main theorem we study the following bifurcation problem

$$\begin{cases} \Delta_p u + \lambda g(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where $\lambda \in \mathbb{R}$ is a positive bifurcation parameter.

The main tools that we use are a radial version for the p-Laplacian of the celebrated global bifurcation theorem due to P. Rabinowitz, a type of Crandall–Rabinowitz result, due to J. García-Melián and J. Sabina de Lis, and bifurcation from infinity.

Problem (1) has been studied by many authors, we point out some of them like [3–6,8,9]. M. del Pino and R. Manásevich studied in [3] the bifurcation phenomena associated with the p-Laplacian operator. As an application, in the radial case, they proved that problem (1) has $n - k + 1$ nontrivial radial solutions assuming that g is continuous, $g(0) = 0$, $g(t)/\varphi_p(t)$ is bounded, and

$$\lim_{t\to 0}\frac{g(t)}{\varphi_p(t)} < \mu_k(p) \leqslant \mu_n(p) < \liminf_{|t|\to\infty}\frac{g(t)}{\varphi_p(t)},$$

where n and k are natural numbers with $k \leqslant n$.

In [4], M. García-Huidobro et al. studied radial solutions of (3) using a technique that extends the results in [3]. In [5], by using blow-up methods, the authors proved the existence of positive radial solutions. On the other hand, S. Liu and S. Li in [8], showed the existence of solutions for problem (1) using Morse Theory via critical groups. Other references for the study of problem (1) can be found in [9].

The paper is organized as follows: In Section 2 we present some important facts. One of them is the global bifurcation theorem in the radial case, which will give us, for each natural k, an unbounded connected component $\mathcal{G}_k$ whose closure contains the bifurcation point $(\mu_k(p)/\lambda_\infty, 0)$. In Section 3 we prove a uniqueness result for the p-Laplacian associated with the radial solutions of (3). In Section 4 we prove Theorem 1.1.

2. Preliminaries

We first note that radial solutions to problem (3) correspond to solutions of

$$\begin{cases} -\big(r^{N-1}\varphi_p(v')\big)' = r^{N-1}\lambda g\big(v(r)\big), & r \in (0, 1), \\ v'(0) = 0 = v(1), \end{cases}$$

where $v(r) = u(x)$, with $r = \|x\|$.

We first consider the radial problem associated to (1), that is,

$$\begin{cases} \big(r^{N-1}\varphi_p\big(u'(r)\big)\big)' + r^{N-1}g\big(u(r)\big) = 0, & 0 < r < 1, \\ u'(0) = 0 = u(1). \end{cases} \tag{4}$$

For $r \in [0, 1]$ we define the *Energy* function associated to (4),

$$\mathcal{E}(r) := \frac{1}{p'}\big|u'(r)\big|^p + G\big(u(r)\big),$$

where $G(\xi) = \int_0^\xi g(t)\,dt$ and $1/p + 1/p' = 1$. Differentiating the function $\mathcal{E}$ and using (4) we get

$$\mathcal{E}'(r) = -\frac{N-1}{r}\big|u'(r)\big|^p \leqslant 0,$$

that is, $\mathcal{E}$ is a decreasing function.

Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(t) := g(t) - \lambda_\infty\varphi_p(t).$$

The function f has the following properties:

(f_0) $f \in C^1(\mathbb{R})$, $f'(t) = o(|t|^{p-2})$ as $t \to 0$, and $f(0) = 0$;

(f_1) $\lim_{t\to 0} \frac{f(t)}{|t|^{p-1}} = 0$;

(f_2) $\lim_{|t|\to\infty} \frac{|f(t)|}{|t|^{q-1}} = 0$, $1 < p \leqslant q < p^*$,

where

$$p^* := \begin{cases} Np/(N-p) & \text{for } p < N, \\ +\infty & \text{for } p \geqslant N. \end{cases}$$

Hence the bifurcation problem (3) is equivalent to

$$\begin{cases} \Delta_p u + \lambda\big[\lambda_\infty \varphi_p(u) + f(u)\big] = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5}$$

with radial version

$$\begin{cases} \big(r^{N-1}\varphi_p(v')\big)' + \lambda r^{N-1}\big[\lambda_\infty \varphi_p(v) + f(v)\big] = 0, \quad 0 < r < 1, \\ v'(0) = 0 = v(1), \end{cases} \tag{6}$$

where $v(r) = u(x)$, with $r = \|x\|$.

Let E be the Banach space of radially symmetric functions in $C^1(\bar{\Omega})$ satisfying $u(x) = 0$ if $\|x\| = 1$. We are interested in classical radial solutions of (5), that is, in pairs (λ, u) with λ in bounded intervals of $(0, +\infty)$ and $v \in E$ such that v is a solution of problem (6).

Lemma 2.1. *If Γ is a connected component in $\mathbb{R} \times E$ of the set of nontrivial solutions to* (6)*, then there exists a positive integer m such that if $(\lambda, v) \in \Gamma$, then v has exactly $m-1$ simple zeroes in $(0, 1)$.*

Proof. Let $(\lambda, v) \in \Gamma$. Therefore $v \in C^1[0, 1]$, $v \neq 0$, and v is a solution of (6). Furthermore we have

$$\big[v(r)\big]^2 + \big[v'(r)\big]^2 > 0 \tag{7}$$

for all $r \in [0, 1]$. This is a consequence of a uniqueness result due to del Pino and Manásevich (see Lemma 5.1 in [3]). Now we prove that v has a finite number of zeroes. Arguing by contradiction, we assume that v has infinite zeroes, thus there exists a sequence of zeroes $\{r_n\} \subset [0, 1]$ such that $r_n \to \bar{r} \in [0, 1]$. By continuity of v we have $v(\bar{r}) = 0$. If r_k and r_{k+1} are two consecutive zeroes, Rolle's Theorem gives us a $\xi_k \in (r_k, r_{k+1})$ with $v'(\xi_k) = 0$. By continuity of v' we get $v'(\bar{r}) = 0$. This implies $v \equiv 0$, which is a contradiction.

If m denote the number of zeroes of v in $[0, 1]$, (7) implies that all of them are simple.

Let Σ_m be the set of pairs $(\lambda, w) \in \Gamma$ such that w has exactly $m-1$ simple zeroes in $(0, 1)$. We shall prove that Σ_m is an open and closed subset of Γ. In fact, let $(\bar{\lambda}, u) \in B_\varepsilon((\lambda, v)) \cap \Gamma$ with $B_\varepsilon((\lambda, v)) \subset \mathbb{R} \times C^1$, where $\varepsilon > 0$ is to be fixed later. In order to prove that Σ_m is open in Γ it is sufficient to show that u has exactly $m-1$ simple zeroes in $(0, 1)$. Let $0 < a_1 < a_2 < \cdots < a_{m-1} < a_m = 1$ be the zeroes of v in $[0, 1]$ and suppose $v(0) > 0$. By continuity of v, it turns out to be positive in $[0, a_1]$. Because the energy function is decreasing it follows that $v(r) \leqslant v(0)$ for $r \in [0, a_1]$. Let

$$b_1 = v(0) = \max_{[0,a_1]} v, \qquad b_2 = \min_{[a_1,a_2]} v, \qquad b_3 = \max_{[a_2,a_3]} v, \qquad \ldots, \quad \text{and} \quad \varepsilon_0 > 0,$$

where $\varepsilon_0 < b/2$, and $b := \min\{|b_i|:\ 1 \leqslant i \leqslant m\} > 0$. Let $C = \{c_1, c_2, \ldots, c_m\}$ be the set of critical points of v. It is easy to check that u and v have the same sign in C, i.e. $u(c_i)v(c_i) > 0$ for all $i = 1, 2, \ldots, m$. By the mean value theorem there exists $\alpha_i \in (c_i, c_{i+1})$ such that $u(\alpha_i) = 0$. Thus, u has at least $m-1$ simple zeroes in $(0, 1)$.

We prove now that u has exactly $m-1$ zeroes in $(0, 1)$. Let $\delta > 0$ be sufficiently small such that for some $\gamma_1 > 0$ and $\gamma_2 > 0$

(i) $|v'(s)| \geqslant \gamma_1$ $\forall s \in \mathcal{C} := [a_1 - \delta, a_1 + \delta] \cup [a_2 - \delta, a_2 + \delta] \cup \cdots \cup [1 - \delta, 1]$;
(ii) $|v(s)| \geqslant \gamma_2$ $\forall s \in \mathcal{D} := [0, 1] \setminus \mathcal{C}$.

Let $\varepsilon < \min\{\frac{1}{2}\gamma_1, \frac{1}{2}\gamma_2, \varepsilon_0\}$. If $s \in \mathcal{D}$,

$$\big|u(s)\big| \geqslant \big|v(s)\big| - \big|v(s) - u(s)\big| \geqslant \gamma_2 - \big|u(s) - v(s)\big| > \gamma_2 - \varepsilon > \gamma_2 - \frac{1}{2}\gamma_2 > 0.$$

Hence u has no zeroes in $\mathcal{D}$. Suppose that there exists another zero $\bar{s}$, different from a_i, such that $\bar{s} \in [a_j - \delta, a_j + \delta]$. By Rolle's Theorem there exists c between a_j and $\bar{s}$ such that $u'(c) = 0$. Thus $|v'(c)| = |v'(c) - u'(c)| < \varepsilon < \frac{1}{2}\gamma_1$, which is a contradiction. Therefore Σ_m is an open subset of Γ.

Finally, we prove that Σ_m is closed. Let $\{(\lambda_n, u_n)\}$ be a sequence in Σ_m such that $(\lambda_n, u_n) \to (\lambda, u) \in \Gamma$. As in our previous proof we can conclude that u has a finite number of zeroes in $(0, 1)$. Since u_n has exactly $m-1$ simple zeroes in (0,1) and $u_n \to u$ in $C^1[0,1]$-norm, u has exactly $m-1$ simple zeroes in (0,1); thus $(\lambda, u) \in \Sigma_m$. This proves that Σ_m is closed in Γ. The proof of the lemma is completed. □

In [3], M. del Pino and R. Manásevich proved the following theorem.

Theorem 2.2. *The set of scalars μ such that* (2) *admits a nontrivial solution consists of an unbounded increasing sequence*

$$0 < \mu_1(p) < \mu_2(p) < \cdots < \mu_k(p) < \cdots.$$

Moreover, the set of solutions of (2) *for $\mu = \mu_k(p)$ is the one-dimensional subspace spanned by a solution ϕ_k of* (2) *with exactly $k-1$ simple zeroes in* $(0,1)$.

Following the main ideas of M. del Pino and R. Manásevich in [3], we prove the next result and for the sake of completeness we sketch its demonstration.

Theorem 2.3. *For each $k \in \mathbb{N}$ there exists a connected component $\mathcal{G}_k \subseteq \mathbb{R} \times C[0,1]$ of the set of nontrivial solutions of problem* (6) *such that its closure $\overline{\mathcal{G}_k}$ contains $(\mu_k(p)/\lambda_\infty, 0)$. Moreover, $\mathcal{G}_k$ is unbounded in $\mathbb{R} \times C[0,1]$, and if $(\lambda, v) \in \mathcal{G}_k$, then v has exactly $k-1$ simple zeroes in* $(0,1)$.

Proof. By the results of M. del Pino and R. Manásevich (see Proposition 4.2 and the proof of Theorem 1.1 in [3]) there exists a component $\mathcal{G}_k \subset \mathbb{R} \times C[0,1]$ whose closure $\overline{\mathcal{G}_k}$ contains $(\mu_k(p)/\lambda_\infty, 0)$ and such that it is unbounded or contains another point $(\mu_i(p)/\lambda_\infty, 0)$ for some $i \neq k$.

It is not difficult to prove that there exists a neighborhood $\mathcal{N}$ of $(\mu_k(p)/\lambda_\infty, 0)$ such that $(\lambda, v) \in \mathcal{N} \cap \mathcal{G}_k$ implies that v has exactly $k-1$ simple zeroes in $(0,1)$.

Let $\mathcal{G}_k = \Gamma$ and Σ_k be as in Lemma 2.1. $\Sigma_k \neq \emptyset$ because $(\mu_k(p)/\lambda_\infty, 0) \in \overline{\mathcal{G}_k}$ implies that there exists a $(\mu, V) \in \mathcal{N} \cap \mathcal{G}_k$ such that V has exactly $k-1$ simple zeroes in $(0,1)$, thus $(\mu, V) \in \Sigma_k$. Since Σ_k is a closed and open set in $\mathcal{G}_k$, we have that $\Sigma_k = \mathcal{G}_k$. Hence $\overline{\mathcal{G}_k}$ is unbounded. The theorem is proved. □

3. A uniqueness result for the p-Laplacian

In this section we will show the existence and uniqueness of solutions to the initial value problem

$$\begin{cases} \left(r^{N-1}\varphi_p(u')\right)' + r^{N-1}g(u) = 0, & 0 < r < 1, \\ u(r_0) = \alpha, & u'(r_0) = \gamma, \end{cases} \tag{8}$$

for $r_0 \in [0,1)$, where α and γ are real numbers.

We observe that u satisfies (8) if and only if u satisfies the integral equation

$$u(r) = \alpha + \int_{r_0}^{r} \varphi_{p'}\left((r_0/t)^{N-1}\varphi_p(\gamma) - \int_{r_0}^{t} (s/t)^{N-1} g\big(u(s)\big)\,ds\right) dt.$$

That is, u is a solution of (8) if and only if u is a fixed point of the operator $T : C[0,1] \to C[0,1]$ defined by

$$(Tu)(r) := \alpha + \int_{r_0}^{r} \varphi_{p'}\left((r_0/t)^{N-1}\varphi_p(\gamma) - \int_{r_0}^{t} (s/t)^{N-1} g\big(u(s)\big)\,ds\right) dt. \tag{9}$$

Theorem 3.1. *Problem* (8) *has a solution $u \in C^1[0,1]$.*

Proof. First we prove local existence over an interval of the form $[r_0, r_0 + \delta_1]$, where $0 < \delta_1 < 1$ is a number to be chosen such that $[r_0, r_0 + \delta_1] \subset (0,1)$.

From (g_3) it follows that $\lim_{t \to 0} \frac{g(t)}{\varphi_p(t)} = \lambda_\infty$. From the definition of λ_∞ and the continuity of the functions φ_p and g, there exists a positive constant C such that $|g(t)| \leqslant C|\varphi_p(t)|$, for all $t \in \mathbb{R}$.

Let T be the operator defined by (9) for $u \in C[r_0, r_0 + \delta_1]$. If $\|u\| := \|u\|_{C[r_0, r_0+\delta_1]}$, then for $r_0 \leqslant r \leqslant r_0 + \delta_1$,

$$\begin{aligned} \big|(Tu)(r)\big| &\leqslant |\alpha| + \int_{r_0}^{r} \varphi_{p'}\left(\left|(r_0/t)^{N-1}\varphi_p(\gamma) - \int_{r_0}^{t} (s/t)^{N-1} g\big(u(s)\big)\,ds\right|\right) dt \\ &\leqslant |\alpha| + \int_{r_0}^{r} \varphi_{p'}\left(|\gamma|^{p-1} + C\|u\|^{p-1}\frac{t^N - r_0^N}{Nt^{N-1}}\right) dt \end{aligned}$$

$$\begin{aligned}
&\leqslant |\alpha| + \int_{r_0}^{r} \left(|\gamma|^{p-1} + C\|u\|^{p-1}|t-r_0|\right)^{p'-1} dt \\
&\leqslant |\alpha| + 2^{p'-1} \int_{r_0}^{r} \left(|\gamma| + (C\delta_1)^{p'-1}\|u\|\right) dt \\
&\leqslant |\alpha| + 2^{p'-1}|\gamma| + (2C)^{p'-1}\delta_1^{p'}\|u\| \\
&= A + (2C)^{p'-1}\delta_1^{p'}\|u\|, \quad \text{where } A := |\alpha| + 2^{p'-1}|\gamma|.
\end{aligned}$$

Since the energy is a decreasing function it follows that $\mathcal{E}(r_0) \leqslant \mathcal{E}(0) = G(d)$, where $d = u(0)$. Thus

$$G(d) \geqslant \frac{1}{p'}\left|u'(r_0)\right|^p + G\left(u(r_0)\right).$$

In addition, by (g_2), it is easy to prove that there exist constants $b > 0$ and $b_1 \in \mathbb{R}$ such that

$$G(s) \geqslant \frac{b}{p}|s|^p + b_1, \quad \text{for all } s \in \mathbb{R}.$$

Then, if $K := \min\{1/(p'2^{p(p'-1)}), b/p\}$, we get

$$\begin{aligned}
G(d) &\geqslant \frac{1}{p'}\left|u'(r_0)\right|^p + \frac{b}{p}\left|u(r_0)\right|^p + b_1 = \frac{1}{p'}|\gamma|^p + \frac{b}{p}|\alpha|^p + b_1 \\
&\geqslant K\left(|\alpha|^p + \left(2^{p'-1}|\gamma|\right)^p\right) + b_1 \\
&\geqslant 2^{1-p}KA^p + b_1.
\end{aligned}$$

Therefore

$$A \leqslant 2^{1/p'}\left[\frac{G(d) - b_1}{K}\right]^{1/p} =: C(d).$$

Thus, the norm of T depends only on the initial data in zero. Let $R > C(d)$, and $\delta_1 < 1$ be such that $(2C)^{p'-1}\delta_1^{p'} \leqslant (R - C(d))/R$. It follows that T maps the closed ball $\overline{B_R(0)} \subset C[r_0, r_0 + \delta_1]$ into itself. The Arzèla–Ascoli Theorem implies that T is a compact operator. By Schauder's fixed point theorem we see that problem (8) has a local solution u. It is clear that $u \in C^1[r_0, r_0 + \delta_1]$.

We now consider problem (8) with initial data $u'(r_0 + \delta_1) = \gamma_1$ and $u(r_0 + \delta_1) = \alpha_1$, and $r_0 + \delta_1 \leqslant r \leqslant r_0 + \delta_1 + \delta_2$, where $0 < \delta_2 < 1$ is the number that gives us a local solution. Following a similar method as above we find that $\delta_2 = \delta_1$. Hence, we can extend the solution to the interval $[r_0, r_0 + 2\delta_1]$. Repeating this argument, in a finite number of steps we extend the solution to the interval $[r_0, 1]$. □

Theorem 3.2. *Problem* (8) *has a unique solution* $u \in C^1[0, 1]$.

Proof. First we prove local uniqueness. By Theorem 3.1 there exists a solution to (8) in $[0, 1]$. Let $\bar{r} \in [0, 1)$ and u_1 and u_2 be local solutions of (8) such that $u_i(\bar{r}) = \alpha$ and $u_i'(\bar{r}) = \gamma$, $i = 1, 2$. We prove that there exists an interval $\bar{V} := [\bar{r}, \bar{r} + \varrho]$ such that $u_1|_{\bar{V}} = u_2|_{\bar{V}}$. From now on the letter C will denote different constants. For $i = 1, 2$, we have

$$\left|\int_{\bar{r}}^{t} (s/t)^{N-1} g\left(u_i(s)\right) ds\right| \leqslant C\|u_i\|^{p-1}\frac{|t^N - \bar{r}^N|}{Nt^{N-1}} \to 0, \quad \text{as } t \to \bar{r}.$$

Hence, if $\bar{r} > 0$, for $i = 1, 2$, we see that

$$\Theta_i(t) := (\bar{r}/t)^{N-1}\varphi_p(\gamma) - \int_{\bar{r}}^{t} (s/t)^{N-1} g\left(u_i(s)\right) ds \to \varphi_p(\gamma), \quad \text{as } t \to \bar{r}. \tag{10}$$

We discuss the case $\bar{r} > 0$ for different values of α and γ. Suppose $\bar{r} > 0$, $\gamma \neq 0$ and $\alpha \in \mathbb{R}$. Thus, $\varphi_p(\gamma) \neq 0$. Since φ_p is C^1 near $\varphi_p(\gamma)$, from the mean value theorem and (10), we see that there are $\varrho_0 > 0$ and $C_1 > 0$ such that, for $r \in [\bar{r}, \bar{r} + \varrho_0]$,

$$\left|u_1(r) - u_2(r)\right| \leqslant C_1 \int_{\bar{r}}^{r} \left|\Theta_1(t) - \Theta_2(t)\right| dt.$$

$g \in C^1$ implies that g is Lipschitz in any bounded interval. Therefore

$$\left|\Theta_1(t) - \Theta_2(t)\right| \leqslant \int_{\bar r}^{t} (s/t)^{N-1}\left|g\big(u_1(s)\big) - g\big(u_2(s)\big)\right| ds \leqslant C\|u_2 - u_1\| \frac{t^N - \bar r^N}{t^{N-1}} \leqslant C\|u_2 - u_1\|$$

which implies that

$$\left|u_1(r) - u_2(r)\right| \leqslant CC_1\varrho_0\|u_1 - u_2\|, \quad \text{for all } r \in [\bar r, \bar r + \varrho_0].$$

Choosing $\varrho_0 > 0$ such that $CC_1\varrho_0 < 1$, we see that $u_1(r) = u_2(r)$ in $[\bar r, \bar r + \varrho_0]$. In a similar way we prove uniqueness over an interval of the form $[\bar r - \varrho, \bar r]$.

In the case $\bar r > 0, \gamma = 0 = \alpha$, it is easy to show that the solution is the trivial one. In fact, since

$$u(r) = -\int_{\bar r}^{r} \varphi_{p'}\left(\int_{\bar r}^{t} (s/t)^{N-1} g\big(u(s)\big)\, ds\right) dt,$$

if $u \not\equiv 0$ in an interval of the form $[\bar r, \bar r + \varrho]$, with ϱ small, then

$$\left|u(r)\right| \leqslant C\|u\| \int_{\bar r}^{r} \varphi_{p'}\left(\frac{t^N - \bar r^N}{t^{N-1}}\right) dt \leqslant C\varrho\|u\|.$$

This implies that $1 \leqslant C\varrho$, which is a contradiction. We observe that this argument also holds when $\bar r = 0$.

Small modifications of the latter arguments gives us uniqueness in case $\bar r > 0$, $\gamma = 0$, $\alpha \neq 0$, and $\alpha \neq \beta$, where $\beta = \beta_i$, $i = 1, 2$.

Now we study the case $\bar r > 0$, $\gamma = 0$, and $\alpha = \beta$. Let $v(r) := u(r) - \beta$, where u is a solution of the differential equation (8) with initial conditions $u(\bar r) = \beta$, $u'(\bar r) = 0$. Then v is solution of

$$\begin{cases} \left(r^{N-1}\varphi_p(v')\right)' + r^{N-1}\tilde g(v) = 0, & 0 < r < 1, \\ v'(\bar r) = 0 = v(\bar r), \end{cases} \tag{11}$$

where $\tilde g(t) = g(t + \beta)$. Therefore,

$$v(r) = -\int_{\bar r}^{r} \varphi_{p'}\left(\int_{\bar r}^{t} (s/t)^{N-1} \tilde g\big(v(s)\big)\, ds\right) dt.$$

Since $|g(t)| \leqslant C|t|^{p-1}$ for $t \in \mathbb{R}$, it follows that $|\tilde g(t)| \leqslant C|t + \beta|^{p-1}$. For t near $\bar r$ we have

$$C \geqslant \left|\frac{\tilde g(t)}{\varphi_p(t+\beta)}\right| = \left|\frac{\tilde g(t)}{\varphi_p(t)}\right| \left|\varphi_p\left(\frac{t}{t+\beta}\right)\right|.$$

Near $\bar r$, the second factor in the previous inequality is bounded from below by a positive constant. Hence there is $\tilde C > 0$ such that

$$\left|\frac{\tilde g(t)}{\varphi_p(t)}\right| \leqslant \tilde C.$$

Repeating the same arguments as above we obtain $v \equiv 0$ in $[\bar r, \bar r + \varrho]$. This shows that $u \equiv \beta$ in such an interval.

Suppose now that $\bar r = 0, \gamma = 0$, and $\alpha = \beta$. By hypothesis

$$\limsup_{t \to \beta} \left|\frac{g(t)}{\varphi_p(t - \beta)}\right| \leqslant C,$$

$|\tilde g(t)| \leqslant C|\varphi_p(t)|$ for t near zero. Hence $v := u - \beta$ satisfies the ordinary differential equation in (11) with $v'(0) = 0 = v(0)$. Thus

$$v(r) = -\int_{0}^{r} \varphi_{p'}\left(\int_{0}^{t} (s/t)^{N-1} \tilde g\big(v(s)\big)\, ds\right) dt,$$

and we have $|v(r)| \leqslant C\varrho\|v\|$. If $v \not\equiv 0$ in an interval of the form $[0, \varrho]$, with ϱ small, then $1 \leqslant C\varrho$ for all $\varrho > 0$, which is a contradiction. Therefore there exists $\varrho > 0$ such that $v \equiv 0$ in $[0, \varrho]$. Thus $u \equiv \beta$ in such an interval. This proves local uniqueness.

In order to show global uniqueness, let U and V be solutions of class $C^1[0,1]$ of (8) in $[0, 1]$, and let $r_1 \in [0, 1]$. By local uniqueness, the problem has a unique solution in an interval of the form $[r_1, r_1 + \varrho]$, which implies $U(r_1) = V(r_1)$. Because r_1 is arbitrary in $[0, 1]$, we see that $U(r_1) = V(r_1)$ for all $r_1 \in [0, 1]$. $\square$

4. Proof of main theorem

For $d \in \mathbb{R}$ and $\lambda > 0$ we will denote by $u(\cdot, \lambda, d)$ the solution to

$$\begin{cases} \big(\varphi_p(u')\big)' + \dfrac{N-1}{r}\varphi_p(u') + \lambda g(u) = 0, & 0 < r < 1, \\ u'(0) = 0, \qquad u(0) = d. \end{cases} \tag{12}$$

Standard arguments on dependence on parameters show that u is a differentiable function in the variable (r, λ, d) (see [2, Theorem 4.5.1]). An elementary calculation shows that if $\alpha > 0$, then

$$u(r/\alpha, \lambda, d) = u\big(r, \lambda/\alpha^p, d\big).$$

Differentiating this relation with respect to α and taking $\alpha = 1$ we obtain

$$ru'(r, \lambda, d) = p\lambda u_\lambda(r, \lambda, d).$$

By using the implicit function theorem and arguing as in the proof of Lemma 2.3 in [1], we can prove the following lemma, which will be used to show that the connected components bifurcating from radial eigenvalues are unbounded in the λ-direction.

Lemma 4.1. *If J is a connected component of $\{(\lambda, d)\colon\ d \neq 0, u(1, \lambda, d) = 0\}$, then there exist an open interval $(a, b) \subset \mathbb{R} \setminus \{0\}$ and a continuous function $h : (a, b) \to (0, \infty)$ such that $(\lambda, d) \in J$ if and only if $d \in (a, b)$ and $\lambda = h(d)$. Moreover, if $a \in \mathbb{R} \setminus \{0\}$, then $\lim_{d\to a} h(d) = \infty$. Similarly, if $b \in \mathbb{R} \setminus \{0\}$, then $\lim_{d\to b} h(d) = \infty$.*

By uniqueness of solutions to the initial value problem it is clear that for λ in bounded intervals of $(0, \infty)$ the problems

$$\begin{cases} \big(r^{N-1}\varphi_p(u')\big)' + \lambda r^{N-1}\big[\lambda_\infty \varphi_p(u) + f(u)\big] = 0, & 0 < r < 1, \\ u'(0) = 0, \qquad u(0) = \beta_i, & i = 1, 2, \end{cases} \tag{13}$$

have a unique solution $u(r) \equiv \beta_i$. Thus if J is as in Lemma 4.1, then the domain of h cannot include β_i.

By Theorem 2.3 we know that for each $k \in \mathbb{N}$, $(\mu_k(p)/\lambda_\infty, 0)$ is a bifurcation point of (6). Moreover, we have

Lemma 4.2. *For each $k \in \mathbb{N}$, $(\mu_k(p)/\lambda_\infty, \infty)$ is a bifurcation point of* (6)*.*

Proof. Suppose that (λ, u), with $u \neq 0$, satisfy (6). Let $v = u/\|u\|^2$. Then v is a solution to

$$\begin{cases} \big(r^{N-1}\varphi_p(v')\big)' + \lambda r^{N-1}\big[\lambda_\infty \varphi_p(v) + \hat f(v)\big] = 0, & 0 < r < 1, \\ v'(0) = 0 = v(1), \end{cases} \tag{14}$$

where

$$\hat f(v) = \begin{cases} \|v\|^{2(p-1)} f(v/\|v\|^2) & \text{if } v \neq 0, \\ 0 & \text{if } v = 0. \end{cases}$$

Condition (f_2) implies that $\hat f \in C(\mathbb{R})$. It is well known that $(\mu_k(p)/\lambda_\infty, \infty)$ is a bifurcation point of (6) if and only if $(\mu_k(p)/\lambda_\infty, 0)$ is a bifurcation point of (14). It is easy to see that $\hat f$ satisfies (f_1) and (f_2). Theorem 2.3 implies that for each $k \in \mathbb{N}$ there is a connected component $\mathcal{G}_k^* \subset \mathbb{R} \times C[0,1]$ of the set of nontrivial solutions of (14) whose closure $\overline{\mathcal{G}_k^*}$ contains $(\mu_k(p)/\lambda_\infty, 0)$. Moreover, $\overline{\mathcal{G}_k^*}$ is unbounded in $\mathbb{R} \times C[0,1]$, and $(\lambda, v) \in \mathcal{G}_k^*$ implies that v possesses exactly $k-1$ simple zeroes in $(0,1)$. This proves that $(\mu_k(p)/\lambda_\infty, \infty)$ is a bifurcation point of (6). □

Let $\mathcal{G}_k \subset \mathbb{R} \times C[0,1]$ be the connected component of nontrivial solutions of (6) which contains $(\mu_k(p)/\lambda_\infty, 0)$. By Theorem 2.3, $\mathcal{G}_k$ is unbounded in $\mathbb{R} \times C[0,1]$, and if $(\lambda, u) \in \mathcal{G}_k$, then u has $k-1$ simple zeroes in (0,1). By the results in [7] (see Theorem 1), there exists $\varepsilon = \varepsilon(k) > 0$ such that for $|s| < \varepsilon$ the problem (12) has solutions of the form $(\lambda_k(s), s[\phi_k + y_k(s)])$, where $\|sy_k(s)\| = o(s)$ and $|\lambda_k(s) - \mu_k(p)/\lambda_\infty| = o(1)$ as $s \to 0$. Similarly, if (λ, u) is a solution to (12) on the branch bifurcating from $(\mu_k(p)/\lambda_\infty, \pm\infty)$, then $u = s(\phi_k + y_k(s))$, where $\|sy_k(s)\| = o(s)$ and $|\lambda_k(s) - \mu_k(p)/\lambda_\infty| = o(1)$ as $s \to \pm\infty$.

We will denote by $\mathcal{O}_k^+$ the connected component of nontrivial solutions to (6) bifurcating from $(\mu_k(p)/\lambda_\infty, 0)$ and containing elements of the form $(\lambda_k(s), s[\phi_k + y_k(s)])$, with $s > 0$. Similarly we define $\mathcal{O}_k^-$. Also we define $\mathcal{I}_k^+$ as the connected component of nontrivial solutions to (6) bifurcating from $(\mu_k(p)/\lambda_\infty, \infty)$ and containing elements of the form $(\lambda_k(s), s[\phi_k + y_k(s)])$, with $s > 0$ sufficiently large. Similarly we define $\mathcal{I}_k^-$.

Lemma 4.3. *Let $J = \{(\lambda, u(0))\colon\ (\lambda, u) \in \mathcal{I}_2^-\}$. If h, a, b are as in Lemma 4.1, then $a = -\infty$ and $b < 0$.*

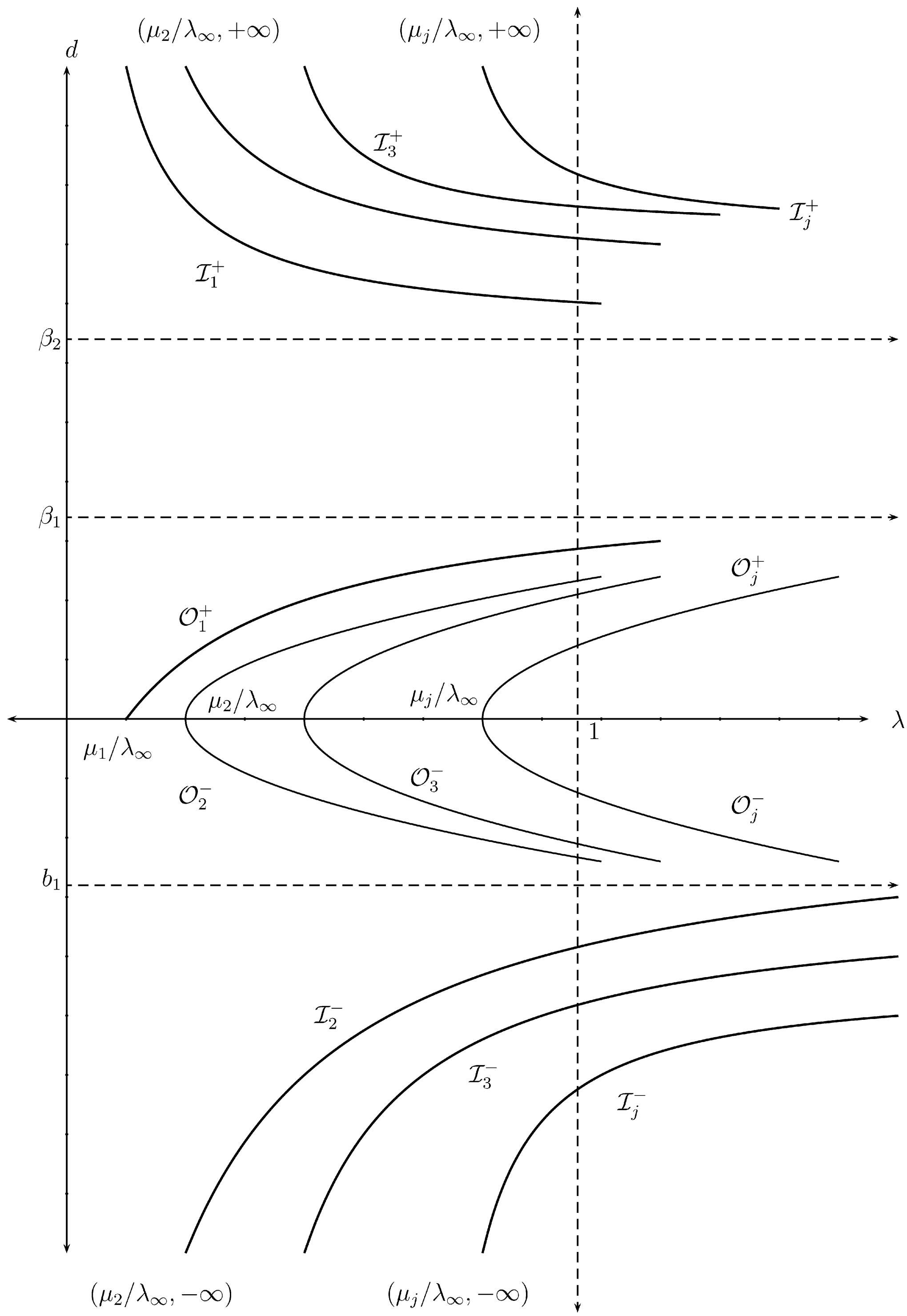

Fig. 1.

Proof. Since $\mathcal{I}_2^-$ contains elements of the form $(\lambda_2(s), s[\phi_2 + y_2(s)])$ with $s < 0$ and $|s|$ large, and $\lambda_2(s)$ is near $\mu_2(p)/\lambda_\infty$, we have $a = -\infty$ and $\lim_{d\to-\infty} h(d) = \mu_2(p)/\lambda_\infty$. Since $\|sy_2(s)\| = o(s)$ as $s \to -\infty$ we have

$$\lim_{s\to-\infty}\bigl[s\phi_2(c) + sy_2(s)(c)\bigr] \to \infty, \tag{15}$$

where c is the critical point of the eigenfunction ϕ_2 in $(0, 1]$.

We claim that the function

$$m(d) := \max\{u(r, h(d), d):\ r \in [0,1]\}$$

is continuous. Let $d_0 \in (a,b)$ and $\{d_n\}$ be a sequence such that $d_n \to d_0$. Of course $h(d_n) \to h(d_0)$. Let $\{t_n\} \subseteq [0,1]$ be such that $t_n \to \bar{t}$, and $m(d_n) = u(t_n, h(d_n), d_n)$. Since $m(d_n) \geqslant u(r, h(d_n), d_n)$, we have

$$\lim_{n\to\infty} m(d_n) = u(\bar{t}, h(d_0), d_0) \geqslant u(r, h(d_0), d_0), \quad \text{for all } r \in [0,1].$$

Thus, $u(\cdot, h(d_0), d_0)$ has a maximum in $\bar{t}$, which is unique because u has exactly one zero in $(0,1)$. Therefore $m(d_n) \to m(d_0) = u(\bar{t}, h(d_0), d_0)$. Which proves that m is a continuous function.

From (15) we see that $\lim_{d\to-\infty} m(d) = +\infty$. If $b = 0$, for small $d < 0$ we obtain that $m(d) = |d|$ and $\lim_{d\to 0} m(d) = 0$. By the intermediate value theorem there exists $d_i < 0$ such that $m(d_i) = \beta_i$, i.e., there exists $t_i \in (0,1)$ with $u(t_i, h(d_i), d_i) = \beta_i$ and $u'(t_i, h(d_i), d_i) = 0$. By uniqueness of solutions to initial value problem we have $u(t, h(d_i), d_i) = \beta_i$ for all $t \in [0,1]$. In particular $d_i = u(0) = \beta_1 = \beta_2$, which contradicts that $d_i < 0$. Thus $b < 0$, which proves the lemma. □

By uniqueness of solutions to (13), $\mathcal{O}_k^+$ does not intersect the line $\|u\| = \beta_1$ and consequently $b < \beta_1$. Similarly, $\mathcal{I}_k^+$ does not intersect the line $\|u\| = \beta_2$, thus $a > \beta_2$. This proves that there is not connection between zero and infinity. More precisely we have the following lemma.

Lemma 4.4. *Let $0 < \beta_1 < \beta_2$ be such that $g(\beta_1) = g(\beta_2) = 0$, $g(t) > 0$ for $t \in (0,\beta_1) \cup (\beta_2, \infty)$. Let (λ, d) be such that $(\lambda, u) \in \mathcal{O}_k^+$. Then $d < \beta_1$. Similarly, for (λ, d) with $(\lambda, u) \in \mathcal{I}_k^+$ we have $d > \beta_2$.*

We observe that $\mathcal{O}_k^+$ and $\mathcal{I}_k^+$ do not intersect the axe $\lambda = 0$ because the equation

$$\begin{cases} (r^{N-1}\varphi_p(u'))' = 0, & 0 < r < 1, \\ u'(0) = 0 = u(1), \end{cases}$$

has only the trivial solution.

Let us denote by h_1 and $b_1 < 0$ the continuous function and the number such that

$$\{(\lambda, u(0)):\ (\lambda, u) \in \mathcal{I}_2^-\} = \{(h_1(d), d):\ d \in (-\infty, b_1)\}.$$

Imitating the proof of Corollary 3.2 in [1], we can prove the following lemmas.

Lemma 4.5. *If $J = \{(\lambda, u(0)):\ (\lambda, u) \in \mathcal{O}_k^-\}$ with $k \geqslant 2$, and a, b are as in Lemma 4.1, then $a \geqslant b_1$.*

Lemma 4.6. *If $J = \{(\lambda, u(0)):\ (\lambda, u) \in \mathcal{I}_k^-\}$ with $k > 2$, and a, b are as in Lemma 4.1, then $b \leqslant b_1$.*

Fig. 1 illustrates the bifurcation diagram for problem (6).

Proof of Theorem 1.1. Let $k \in \{1, 2, \dots, j\}$ and $J = \{(\lambda, u(0)):\ (\lambda, u) \in \mathcal{O}_k^+\}$. Let (a,b) and h be as in Lemma 4.1. By the definition of $\mathcal{O}_k^+$ we see that $a = 0$ and $\lim_{d\to 0} h(d) = \mu_k(p)/\lambda_\infty < 1$. By Lemma 4.1 $\lim_{d\to b} h(d) = +\infty$. By the intermediate value theorem there exists $d_k \in (a,b)$ such that $h(d_k) = 1$. Thus $(1, u(\cdot, 1, d_k)) \in \mathcal{O}_k^+$ is a radial solution to (1).

Similarly, if $J = \{(\lambda, u(0)):\ (\lambda, u) \in \mathcal{O}_k^-\}$ with $k \in \{2, 3, \dots, j\}$ and (a,b) and h are as in Lemma 4.1, then $a \geqslant b_1$, and $\lim_{d\to a} h(d) = +\infty$, and there exists $\delta_k \in (a, 0)$ such that with $h(\delta_k) = 1$. Thus $(1, u(\cdot, 1, \delta_k)) \in \mathcal{O}_k^-$ is a radial solution of (1).

Similar arguments show that $\mathcal{I}_k^-$ for $k = 2, 3, \dots, j$, and $\mathcal{I}_k^+$ for $k = 1, 2, \dots, j$, contain a radial solution of (1). This proves Theorem 1.1. □

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